Criticism of Feynman's analysis of the ratchet as an engine

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Abstract

The well-known discussion on an engine consisting of a ratchet and a pawl in [R.P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics, vol. 1* (Addison Wesley, Reading, Massachusetts, 1963), pp. 46.1-46.9] is shown to contain some misguided aspects: since the engine is *simultaneously* in contact with reservoirs at different temperatures, it can never work in a reversible way. As a consequence, the engine can never achieve the efficiency of a Carnot cycle, not even in the limit of zero power (infinitely slow motion), in contradiction with the conclusion reached in the *Lectures*.

1 Introduction

Chapter 46 of *The Feynman Lectures on Physics* [1] contains a celebrated illustration of the impossibility of obtaining work from thermal fluctuations with an efficiency greater than that of a Carnot cycle. A careful analysis of a device that, at first sight, seems to lift a weight using the thermal energy of a gas, reveals that there exists in fact a dissipation which prevents the failure of the second law of thermodynamics. The device is nothing but an axle with vanes in one of its ends and a ratchet in the other that, in principle, can move only in one direction (Fig.1). If the vanes are surrounded by a gas at a given temperature, they will undergo collisions with the molecules of the gas and oscillate as a one-dimensional Brownian rotor. However, due to the presence of the ratchet at the other end of the axle, only fluctuations in one direction, if they are strong enough, could make the ratchet lift the pawl and advance to the next notch [2].

Feynman carried out an analysis of such an engine proving that, in order to obtain work out of thermal fluctuations, the vanes must be within a thermal bath at a temperature T_1 higher than the temperature T_2 of the ratchet. Moreover, he calculated, under some simplifying assumptions, the efficiency of the engine and found it equal to that of a Carnot cycle. This example and the corresponding analysis, beside its pedagogical interest, is cited as a proof of the impossibility of an automatic device acting as a Maxwell demon [3] and has been also inspiration of a currently very active research field on transport induced by Brownian motion in asymmetric potentials [4].

We point out in this paper a misconception of Feynman's analysis which, from our point of view, diminishes its pedagogical virtues. Feynman's analysis focuses on an ideal situation in which the device is supposed to work in a reversible way, so Carnot efficiency is reached. This ideal situation corresponds to the limit of very slow motion of the engine, i.e., to a *quasistatic*

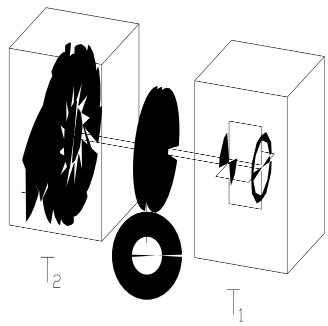

Figure 1: The ratchet and pawl machine (after [1]).

process. We claim that such quasistatic process is not *reversible*. The reason is that, in the ratchet engine of Fig.1, the system is in contact *simultaneously* with the two thermal baths at different temperatures. Consequently, it cannot be in thermal equilibrium and an irreversible heat transfer unavoidably occurs.

It should be emphasized that we do not refer to a heat transfer through the *materials* of the elements of the ratchet engine. These materials can be considered perfect isolators. In Feynman's discussion the vanes and the ratchet are mechanically linked, but are thermally isolated. This would be a pertinent pedagogical idealization. However, in this paper we show that the mechanical link between the vanes and the ratchet necessarily implies that the thermal baths are *not* thermally isolated. There is an essential incompatibility between mechanical coupling and thermal isolation because the mechanical coupling induces, via fluctuations, a heat transfer between the thermal baths.

Notice the difference between the Carnot and the ratchet engines. In a Carnot cycle the heat transfer can ideally occur in a reversible way because the engine is never simultaneously in contact with the two thermal baths. The steps where the heat transfer takes place, the isothermal expansion and compression, can be considered approximately reversible if they are slow enough or, more precisely, much slower than the relaxation to equilibrium of the gas in the engine. For the ratchet this is not the case. The system is simultaneously in contact with the two thermal baths, so the heat transfer cannot be performed reversibly, not even in the quasistatic limit. There are no two time scales, as in the Carnot cycle, to compare.

The paper is organized as follows: in section II we briefly review Feynman's analysis and in section III we discuss our criticism in detail. We first recall that Carnot efficiency is equivalent to zero entropy production and then we analyze the stationary regime that Feynman assumes to be reversible, arguing its intrinsic irreversibility on the grounds of general principles from Statistical Mechanics. Finally, we present our conclusions in section IV.

2 Sketch of Feynman's analysis

Consider first the setup of Fig.1 without any weight. Feynman convincingly argues that the engine cannot work if the vanes and the ratchet are at the same temperature. Let T be this

Forward:	Needs energy	$\epsilon + L\theta$	from vane. Rate $= \frac{1}{\tau} e^{-(L\theta+\epsilon)/k_B T_1}$
	Takes from vane	$L\theta + \epsilon$	
	Does work	$L\theta$	
	Gives to ratchet	ϵ	
Backward:	Needs energy	ϵ	from pawl. Rate $= \frac{1}{\tau} e^{-\epsilon/k_B T_2}$
	Takes from ratchet	ϵ	
	Releases work	$L\theta$	} same as above with sign reversed
	Gives to vane	$L\theta + \epsilon$	

Table 1: Summary of operation of ratchet and pawl (from [1]).

temperature and ϵ the energy required to lift the pawl just above the tooth against the spring that pulls it down. For low temperatures, the rate at which a fluctuation provides the vanes with energy enough to move the ratchet to the next tooth can be approximated by the Arrhenius factor, i.e., is proportional to $e^{-\epsilon/k_B T}$. But the pawl itself is also embedded in a thermal bath at temperature T, so it can be lifted by fluctuations from this bath and, moreover, these backward jumps occur at the same rate. Therefore, if both baths are at the same temperature no systematic motion of the ratchet occurs.

Feynman then supposes different temperatures $T_1 > T_2$ for the thermal baths, i.e., the pawl to be colder than the vanes. Now the rates of jumps are no longer equal and this drift can eventually be used to lift the weight. Indeed, there is a value of the weight L_0 such that both rates are equal and the ratchet does not undergo any systematic motion. Assuming again that the rates are proportional to the Arrhenius factor with the same proportionality constant, this value L_0 is easily calculated:

If $L\theta$ is the potential energy [7] the weight L gains when the ratchet performs a *forward* jump (forward direction being the expected direction of motion of the ratchet), then $\epsilon + L\theta$ is the energy needed for such a forward jump. This energy is mainly obtained from the vanes, so the rate of forward jumps is proportional to $e^{-(L\theta+\epsilon)/k_B T_1}$. For a backward jump the energy required is ϵ and Feynman assumes that this energy is taken from the ratchet bath, so the corresponding rate is proportional to $e^{-\epsilon/k_B T_2}$. There is a weight L_0 for which both rates are equal:

$$\frac{L_0\theta + \epsilon}{\epsilon} = \frac{T_1}{T_2}. \tag{1}$$

Let us now turn to the evaluation of the energy transfer between the baths and the ratchet and vanes. We have seen that in a forward jump the system takes an energy $\epsilon + L\theta$ from bath 1. After the jump, an energy ϵ has been dissipated. Feynman assumes that this energy is entirely dissipated to bath 2. In a backward jump, the energy ϵ is taken from bath 2 and after the jump an energy $\epsilon + L\theta$ has been dissipated. The further assumption is that this energy is dissipated to bath 1. Table 1, which is a partial reproduction of Table 46-1 in [1] summarizes the energy transfer for both types of jump.

If L is now chosen to be smaller but close to L_0, then the wheel will move forwards very slowly, lifting the weight. With the above assumptions on the energy transfer, it is not difficult to calculate the efficiency of the engine. If the ratchet performs N_+ forward jumps and N_- backward jumps, the total work done is $(N_+ - N_-)L\theta$ and the amount of heat taken from bath 1 is $(N_+ - N_-)(L\theta + \epsilon)$. Therefore the efficiency is

$$\eta = \frac{L\theta}{L\theta + \epsilon} \tag{2}$$

and, in the limit $L \to L_0$ (or zero power), the efficiency converges to that of a Carnot cycle (see Eq. (1)):

$$\eta \to \eta_c = 1 - \frac{T_2}{T_1}. \tag{3}$$

3 The criticism

Carnot efficiency is reached when an engine works between two baths at different temperatures $T_1 > T_2$ in a reversible way. If in a given period of time the engine takes an amount of heat Q_1 from bath 1, releases Q_2 to bath 2, and performs work $W = Q_1 - Q_2$, ending in its initial state, the only entropy variations in the universe are those of the thermal baths: bath 1 decreases its entropy by Q_1/T_1 and bath 2 increases its by Q_2/T_2. Reversibility implies that the entropy must remain constant. Then:

$$\Delta S = \frac{Q_2}{T_2} - \frac{Q_1}{T_1} = 0 \tag{4}$$

and Carnot efficiency $W/Q_1 = \eta_c$ immediately follows. Any irreversibility, i.e., any finite entropy production $\Delta S > 0$, will reduce the efficiency of the engine.

Therefore, Feynman's calculation implies that the ratchet engine works in a reversible way, i.e. $\Delta S = 0$, when L is infinitely close to L_0. A look at Table 1 gives us the explanation of such reversibility. The energy transfer between the engine and the baths in a forward jump is exactly the same as in the backward jump reversing the signs. Consequently, if a forward jump is followed by a backward one, the net flow of energy is zero. No heat is taken from bath 1, released to bath 2, no work is done on or by the weight. When $L = L_0$ the rate of jumps is the same in each direction, thus the situation is completely reversible.

However, as stressed in the introduction, the ratchet is a system subject to nonequilibrium constrains: different parts are simultaneously in contact with thermal baths at different temperatures. Consequently, the system can never be in thermal equilibrium. The nonequilibrium nature of the state of a system which is in contact with two thermal baths at different temperatures shows up by means of an irreversible heat conduction. On the other hand, notice that, if Table 1 were correct, in the stationary regime $L = L_0$ the two baths at different temperatures do not exchange energy. In other words, the thermal conductivity of the engine —vanes, axle, ratchet, and pawl— would be identically zero. As it will be clear below, here we do not refer to a heat conduction through the materials of the elements of the ratchet engine but through the very degree of freedom that allows the engine to work.

It is not an easy task to estimate the thermal conductivity of the ratchet engine. In the Appendix we calculate the conductivity of simpler but related systems, as a rigid or flexible axle with vanes at both ends. We use a Langevin approach to deal with these examples. In this approach one can see that the energy transfer between a system and a bath consists of two terms: one due to fluctuations (flow from the bath to the system) and the other due to dissipation (flow from the system to the bath). These two terms cancel each other if the state of the system is the Gibbsian equilibrium state e^{-H/k_BT}. However, if a single system is coupled to two thermal baths at different temperatures, its state is no longer e^{-H/k_BT} (which T would we write?) and deviations from equilibrium imply that fluctuation-dissipation balance no longer holds. A net energy flow from the hotter bath to the system and from this to the colder bath occurs. This energy flow is in the form of incoherent fluctuating motion of the mechanical link between both thermal baths, i.e., in the form of heat. It should be emphasized that heat can be transferred through a single degree of freedom. In the Appendix we consider two simple systems and show that they act, as should be expected, as heat conductors, having a non zero thermal condutivity.

For the ratchet engine, we conclude that the stationary regime, $L = L_0$, is an irreversible situation: no work is done but nevertheless a flow of heat goes from the hotter to the colder bath through the mechanical link between both baths. If k is the conductivity of the engine in this stationary regime, then heat flows from the hot to the cold bath at a rate $\dot{Q} = k(T_1 - T_2)$ and entropy is produced at a rate $\Delta\dot{S} = \dot{Q}(1/T_2 - 1/T_1)$. This non zero production of entropy prevents the system to posses the Carnot efficiency. In fact, in the quasistatic limit, $L = L_0$, the efficiency of the engine

$$\eta = \frac{W}{Q_1} \tag{5}$$

obviously vanishes for W goes to zero whereas Q_1 remains finite. Notice that in this respect the ratchet engine also differs from many of the irreversible engines considered by the *Finite Time Thermodynamics* [5] which achieve Carnot efficiency at zero power but smaller efficiencies at finite power (cfr. the celebrated Curzon-Ahlborn formula for the efficiency of an engine working at maximum power [6]).

Let us go back to Table 1 in order to find out which are the less convincing estimations that it contains. We find the last row of Table 1 extremely doubtful. One could conceive limiting situations agreeing with the rest of the entries in the table: the Arrhenius factor is a good approximation for low enough temperatures [8, 9] and, if T_1 is much larger than T_2, the energy $\epsilon + L\theta$ for a forward jump will be mostly taken from the vanes and the excess ϵ mostly dissipated to the ratchet. But, why is there no dissipation to the ratchet bath in a backward jump? In the first page of the chapter it is said: "... an essential part of the irreversibility of our wheel is a damping or deadening mechanism which stops the bouncing [of the pawl]" (here "irreversibility" stands for the asymmetric behavior of the ratchet and it has nothing to do with thermodynamical irreversibility). This damping mechanism could take place in the collisions between the ratchet and the pawl and/or because both are embedded in a gas. In any case, when the pawl is going down in a backward jump, undergoing both the force of its spring and the force of the hanging weight, a damping occurs in the ratchet-and-pawl end of the axle as well as in the vanes end; even the damping will be greater in the former if we consider situations where the vanes are much hotter than the ratchet, $T_1 \gg T_2$.

The last entrance of Table 1 should be replaced in the following way: in a backward jump a part of the excess of energy, say $\gamma(L\theta + \epsilon)$ with $0 < \gamma < 1$, is dissipated to the vane and the rest, $(1-\gamma)(L\theta + \epsilon)$ is dissipated to the ratchet. Then, in the stationary regime, the flow of heat is:

$$\dot{Q} = \dot{N}_+(1-\gamma)(L\theta + \epsilon) = \dot{N}_+(1-\gamma)\epsilon\frac{T_1}{T_2} \tag{6}$$

where $\dot{N}_+$ is the number of forward jumps per unit of time.

There is another important objection that could be made to Feynman's analysis. The assumption that the constant $1/\tau$ in front of the Arrhenius factor is the same for the two thermal baths is not completely justified. These constants, say $1/\tau_1$ and $1/\tau_2$, depend on the detailed structure of each bath and it is possible to conceive situations where they are different. Nevertheless, if the entrances in Table 1 were correct, one could obtain efficiencies bigger than η_c in the case $1/\tau_1 > 1/\tau_2$. Since there is no indication of how Table 1 should be modified in order to deal with this case, Feynman's analysis is a rather incomplete proof of the compatibility of the ratchet engine and the second law.

The answer to this objection lies again in the above modification of Table 1. The relationship between γ, $1/\tau_1$, and $1/\tau_2$ should yield to an efficiency compatible with the second law. This relationship is however hard to find and depends on details of the baths and the coupling between the system and the baths.

4 Conclusions

To conclude, let us stress that our criticism is not only focused on quantitative aspects of Feynman's analysis but it reveals that it is in contradiction with two fundamental facts from Thermodynamics and Statistical Mechanics:

1. Carnot efficiency follows from a zero entropy production, as explained in sec. 2.

2. A system simultaneously coupled to two thermal baths at different temperatures cannot be in thermal equilibrium and therefore it cannot undergo a reversible process.

The above deviation from equilibrium, as indicated by the examples worked out in the Appendix, implies a continuous transfer of heat from the hotter bath to the colder one, i.e., a non zero thermal conductivity and consequently a production of entropy. This thermal conductivity is missing in Table 1.

The misconception in Feynman's analysis is specially relevant from a pedagogical point of view, for it does not contribute to clarify under which conditions a process is thermodynamically reversible and mixes up the concept of reversible process and quasistatic process. Reversibility necessarily implies that the system is in equilibrium at every time during the process. A quasistatic process is usually a reversible one because the constrains of the system are moved much slower than the relaxation of the system to equilibrium, so it can be considered in equilibrium with its constrains at every time. This is the case in a Carnot cycle. However, if the constrains themselves are of a nonequilibrium nature, as, for instance, those of a gas in a box with each side at a different temperature or pressure, a quasistatic process is not a reversible one. In fact, even when the constrains do not change in time, the system is not at equilibrium, as it happens in the ratchet engine.

5 Acknowledgements

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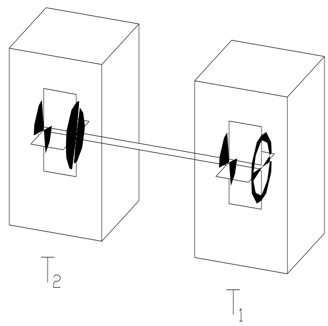

Figure 2: The axle with two vanes.

Appendix

In this Appendix we calculate the thermal conductivity of two simple models related with the ratchet engine. Both consist in a system coupled to two thermal baths at different temperatures.

One degree of freedom.

Consider an axle with vanes at *both* ends. Each end is embedded in a thermal bath at temperatures T_1 and T_2 , respectively (see Fig.2). The angle of the axle, x, performs a Brownian motion due to fluctuations induced by the baths. We will obtain the equation of motion for x and from this equation the energy transfer between the two baths.

Recall that the position of a Brownian particle at temperature T in a potential $V(x)$ is well described by the Langevin equation:

$$m\ddot{x} = -V'(x) - \lambda\dot{x} + \xi(t) \tag{7}$$

where λ is the friction coefficient and $\xi(t)$ is a Gaussian white noise with zero average and temporal correlation

$$\langle \xi(t)\xi(t')\rangle = 2\lambda k_B T \delta(t-t') \tag{8}$$

k_B being the Boltzmann constant. The Fokker-Planck equation for the probability density $\rho(x,p;t)$ of the position x and the momentum $p = m\dot{x}$ corresponding to the Langevin equation (7) reads [10]

$$\frac{\partial}{\partial t}\rho(x,p;t) = \left[-\frac{\partial}{\partial x}\frac{p}{m} + \frac{\partial}{\partial p}\left(V'(x) + \frac{\lambda}{m}p\right) + \lambda k_B T\frac{\partial^2}{\partial p^2}\right]\rho(x,p;t). \tag{9}$$

The stationary solution of (9) is precisely the thermal equilibrium Gibbs state

$$\rho^{(st)}(x,p) = \frac{1}{Z}\exp\left\{-\frac{1}{k_B T}\left[\frac{p^2}{2m} + V(x)\right]\right\}. \tag{10}$$

Using the Fokker-Planck equation (9), one can derive the following evolution equation for the average energy of the particle:

$$\frac{d}{dt}\left\langle \frac{p^2}{2m} + V(x) \right\rangle = \frac{\lambda}{m}\left[k_B T - \frac{\langle p^2 \rangle}{m}\right]. \tag{11}$$

The second term of the r.h.s. is the energy dissipated by the particle (per unit of time) into the bath, whereas the first one is the energy the system takes from the bath due to fluctuations. They of course cancel each other at thermal equilibrium, as it can be easily seen from the equipartition theorem: $\langle p^2 \rangle / m = k_B T$.

Let us consider now the system of Fig. 2. The Langevin equation for the angle x is (for t large enough [11])

$$m\ddot{x} = -\lambda \dot{x} + \xi_1(t) - \lambda \dot{x} + \xi_2(t) \tag{12}$$

where we have assumed the friction coefficient to be equal for the two baths. Now the white Gaussian noises are characterized by the following temporal correlations:

$$\langle \xi_i(t) \xi_j(t') \rangle = 2\delta_{ij} \lambda k_B T_i \delta(t - t') \qquad i, j = 1, 2. \tag{13}$$

The stationary solution corresponding to (12) still can be easily found: it is the equilibrium Gibbs state at temperature $T_{\text{eff}} = (T_1 + T_2)/2$.

The evolution equation for the average energy of the system now reads

$$\frac{d}{dt}\left\langle \frac{p^2}{2m} \right\rangle = \frac{\lambda}{m}\left[k_B T_1 - \frac{\langle p^2 \rangle}{m}\right] + \frac{\lambda}{m}\left[k_B T_2 - \frac{\langle p^2 \rangle}{m}\right], \tag{14}$$

and, by analogy with Eqn. (11) and the interpretation following it, we can identify

$$\dot{Q}_i = \frac{\lambda}{m}\left[k_B T_i - \frac{\langle p^2 \rangle}{m}\right] \tag{15}$$

as the energy flow from bath i to the system. In the stationary regime the whole r.h.s. of (14) is zero but not each term $\dot{Q}_i$ separately, indicating that there is a continuous flow of heat from one bath to the other. Assuming $T_1 > T_2$, heat flows from bath 1 to bath 2 at a rate

$$\dot{Q}_1 = -\dot{Q}_2 = \frac{\lambda}{m}[k_B T_1 - k_B T_{\text{eff}}] = \frac{\lambda k_B}{2m}(T_1 - T_2) \tag{16}$$

as it can be immediately derived by computing $\langle p^2 \rangle$ with the Gibbs ensemble at temperature T_{eff}. Therefore, we see that there is a heat transfer obeying a Fourier law and that the thermal conductivity of the axle is $\lambda k_B/(2m)$.

The heat transfer takes place because the dispersion of p no longer satisfies the equipartition theorem. The system reaches a stationary state which does not correspond to thermal equilibrium. In this particular case, this state has the form of a Gibbs state with an effective temperature but, in general, the form of the stationary probability distribution could be completely different, as in our next example.

Two degrees of freedom. Linear case.

Let us assume now that the axle of the previous example is not rigid but it has a finite torsion coefficient κ. The system now must be described by two degrees of freedom x_1, x_2 corresponding to the angle of the vanes at each end of the axle, respectively. The equations of motion are now

$$m\ddot{x}_1 = -\lambda\dot{x}_1 - \kappa(x_1 - x_2) + \xi_1(t) \tag{17a}$$

$$m\ddot{x}_2 = -\lambda\dot{x}_2 + \kappa(x_1 - x_2) + \xi_2(t), \tag{17b}$$

and the noise correlations are again given by (13). Note that the same equations apply for a dumbell with each mass immersed in a thermal bath at a different temperature.

The evolution equation for the average energy of the system reads

$$\frac{d}{dt}\left\langle \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{\kappa(x_1-x_2)^2}{2} \right\rangle = \frac{\lambda}{m}\left[k_B T_1 - \frac{\langle p_1^2\rangle}{m}\right] + \frac{\lambda}{m}\left[k_B T_2 - \frac{\langle p_2^2\rangle}{m}\right], \tag{18}$$

and, as in the previous example, we can identify the first term of the r.h.s. as the energy transfer rate $\dot{Q}_1$ from bath 1 to the system, whereas the second one is the heat transfer rate $\dot{Q}_2$ from bath 2. Again, in the stationary regime the whole r.h.s. is zero, $\dot{Q}_1 = -\dot{Q}_2$, but not each term separately indicating that there is a continuous flow of heat from one bath to the other.

The calculation of $\langle p_1^2\rangle$ and $\langle p_2^2\rangle$ in the stationary regime is now more involved. We present the main steps of the calculation. Let $r = x_1 - x_2$. From the Langevin equations (17) or its corresponding Fokker-Planck equations, one can get the following evolution equations:

$$m\frac{d}{dt}\langle r^2\rangle = 2\langle r(p_1 - p_2)\rangle \tag{19a}$$

$$\frac{d}{dt}\langle p_1^2\rangle = -\frac{2\lambda}{m}\langle p_1^2\rangle - 2\kappa\langle p_1 r\rangle + 2\lambda k_B T_1 \tag{19b}$$

$$\frac{d}{dt}\langle p_2^2\rangle = -\frac{2\lambda}{m}\langle p_2^2\rangle + 2\kappa\langle p_2 r\rangle + 2\lambda k_B T_2 \tag{19c}$$

$$\frac{d}{dt}\langle p_1 p_2\rangle = -\frac{2\lambda}{m}\langle p_1 p_2\rangle + \kappa\langle r(p_1 - p_2)\rangle \tag{19d}$$

$$m\frac{d}{dt}\langle r(p_1 + p_2)\rangle = \langle p_1^2\rangle - \langle p_2^2\rangle - \lambda\langle r(p_1 + p_2)\rangle \tag{19e}$$

In the stationary regime the above time derivatives are zero. One thus obtains:

$$\langle p_1^2\rangle + \langle p_2^2\rangle = m k_B (T_1 + T_2) \tag{20a}$$

$$\langle p_1^2\rangle - \langle p_2^2\rangle = \frac{m k_B}{1+\alpha}(T_1 - T_2) \tag{20b}$$

where $\alpha = \kappa m/\lambda^2$. The second moment of the momentum

$$\langle p_1^2\rangle = \frac{m k_B}{2(1+\alpha)}\left[(2+\alpha)T_1 + \alpha T_2\right] \tag{21}$$

is again $m k_B$ times a weighted average of the two temperatures. If $\alpha = 0$ this average is T_1, that is, x_1 and x_2 do not interact and each one is in equilibrium with its corresponding bath. On the other hand, for $\alpha \to \infty$ we recover the result obtained above for the rigid axle.

Finally, the heat flow reads

$$\dot{Q}_1 = -\dot{Q}_2 = \frac{\lambda k_B \alpha}{2m(1+\alpha)}(T_1 - T_2) \tag{22}$$

i.e., we obtain again the Fourier law with a positive conductivity $\lambda k_B \alpha/[2m(1+\alpha)]$. Notice that the stationary state cannot be written as proportional to $e^{-H/k_B T_{\text{eff}}}$ for some effective temperature, since, for instance, r and p_i are correlated if $T_1 \neq T_2$.

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[7] We follow Feynman's notation where L is not actually the weight but the torque acting on the wheel and θ the angle between two teeth of the ratchet. In any case, the only relevant quantity is the potential energy $L\theta$.

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[11] A rigorous derivation of this equation shows that a term depending on initial conditions is present [9]. This term is important as far as transient behaviour is concerned and, in particular, the relaxation of a system coupled to a single bath at temperature $T_{\rm eff} = (T_1 + T_2)/2$ is different from that of a system couple to two baths at temperatures T_1, T_2. We will use Eqn. (12) in order to compute steady state properties only and therefore we can safely neglect this initial condition term.